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THESIS

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An Analysis of Mean Absolute Deviation Variability

by

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ABSTRACT

Prediction of demand and setting reorder levels by exponential smoothing and mean absolute deviation is studied for an inventory system. The relation between the standard deviation and mean absolute deviation of demand distribution is discussed. Asymptotic first and second moments of the exponentially-smoothed forecast errors (MAD) are derived. Results obtained from simulation of several normal systems indicated that the smoothing technique is inferior to the classical maximum likelihood estimation method.

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TABLE OF SYMBOLS AND ABBREVIATIONS

x_i	Demand during the i^{th} time period
α	Exponential smoothing constant
$S_t(x)$	Forecasted demand during the i^{th} period
ρ	Correlation coefficient
ψ	Risk level
Δ	Mean absolute deviation of a probability distribution
μ	Mean demand
V( )	Variance of the argument
E( )	Expected value of the argument
e_t	t^{th} period forecast error
$\hat{\sigma}$	Estimate of the standard deviation of the demand by maximum likelihood estimates
$\tilde{\sigma}$	Estimate of the standard deviation of the demand distribution by mean absolute deviation

I. INTRODUCTION

In recent years the technique of forecasting time series by exponential smoothing has received a considerable amount of attention in the literature. As a forecasting technique in inventory theory, it is particularly suited to the situation where demands are time dependent and correlated for much of its intuitive appeal rests on weighting the past in such a way that the most distant past receives less and less weight. When demands are not so related, smoothing may or may not be a suitable forecasting technique. In particular, when demands are independent and from a common distribution such as for the so-called fast moving items, more classical methods like maximum likelihood may be preferable because of the optimal properties that are often demonstrable. Even in such circumstances, however, exponential smoothing is still a candidate for a forecasting scheme and it may very well be that, while not optimal , the difference in such criteria as stockout risk for example, may be negligible compared to optimal procedures from a practical point of view. In that case, forecasting rules need not be changed for items differing in demand pattern and a certain uniformity may thus be achieved.

The main purpose of this thesis is to investigate a model of an inventory system used by the Naval Supply

System Command (NAV SUP) following Brown [Ref. 1]. The model, as described by Ref. 1 and Ref. 5, is briefly as follows:

Single exponential smoothing is an operator given in Ref. 1 as

$$S_t(x) = \alpha \sum_{k=0}^{t-1} \beta^k X_{t-k} + \beta^t X_o \tag{1-1}$$

where x_i is the total demand of the i^{th} period

α is the smoothing constant with $0 < \alpha < 1$

$\beta = 1 - \alpha$

Intuitively speaking the smoothing operator is a method of forecasting by weighting past demands in such a way that the most recent demand is weighted by α and the remaining demands are given less and less weight as time progresses.

The standard deviation, σ, of the demand distribution, which is needed to set the reorder levels, is estimated by the following three statistics:

$e_t = X_t - S_{t-1}(x)$, t^{th} period forecast error,

$$\hat{\Delta} = \alpha \sum_{k=0}^{t-1} \beta^k \left| e_{t-k} \right| + \beta^t \left| e_o \right| \tag{1-2}$$

estimate of the mean absolute deviation of forecast errors, and

$$\tilde{\sigma} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{2-\alpha}{2}}\, \hat{\Delta} \quad ,$$

estimate of the standard deviation of the demand distribution.

This estimate of σ, as explained in Ref. 1, is based on the mean absolute deviation (MAD), Δ, of the forecast errors. Reorder levels, based on the predetermined risk level ψ (defined to be the probability of running out of stock), are found as follows:

$$\psi = P(X > \mu + k\sigma)$$

Here, μ is the mean of X, and k is a constant which can be determined from the probability tables for a given risk level, ψ. The reorder level is then given by

$$\text{Reorder Level} = \mu + k\sigma \tag{1-3}$$

where k is the given above.

In the analysis this model was slightly modified because of the following considerations:

1. X_o, the demand asked from the inventory system before it is placed in operation, is zero. Although one may predict a number for a fictitious inventory, there will be no observation for X_o. Hence e_o is zero.

2. α^N and β^N can be assumed to be approximately equal to zero for values of N > 30, if $0.01 \le \alpha \le 0.99$. In view of this, the modified form of (1-1) and (1-2) used in the analysis was

$$S_t(x) = \alpha \sum_{k=0}^{t-1} \beta^k X_{t-k} \tag{1-4}$$

$$\hat{\Delta} = \alpha \sum_{k=0}^{t-1} \beta^k \left| e_{t-k} \right| \tag{1-5}$$

In this model, not only is demand forecasted by exponential smoothing but demand variability is estimated by means of exponentially smoothed absolute deviations. Such estimates are in turn used to set safety levels for inventory order rules.

The model described above has been subjected to many criticisms. Zehna [Ref. 7] has indicated that the estimate of standard deviation might be very poor depending upon the underlying demand distribution. Based on the results obtained by Zehna [Ref. 8], the analysis in this paper takes on its starting point on investigation of the ratio Δ/σ, a key element in the estimation of σ. It is quite evident that any weakness in the estimation procedure for σ affects the reorder levels. An underestimate imposes higher risk and an overestimate causes unnecessary expenditures. In particular, the resulting consequences could be catastrophic in a military inventory system.

It is quite evident that the probability distributions of certain key statistics would be desirable in any extention of the existing analysis. Although some of the distributions required have been derived, the distribution of $\hat{\Delta}$, a most important statistic, could not be derived explicitly. However, the first and second asymptotic moments of $\hat{\Delta}$ were obtained, and it is shown that the asymptotic variance of $\hat{\Delta}$ is finite for the case where the demand has a normal distribution. Therefore, the next step of the analysis used simulation techniques to gain an insight into the behavior of the distribution of $\hat{\Delta}$, and to observe the effects of a finite variance. Reorder levels and their dispersions along with the risk levels attained are analyzed. The resulting errors due to the variability of safety levels is examined in detail and the consequences are evaluated from several points of view when demands are normal. Moreover the results are compared to maximum likelihood methods establishing that a considerable difference exists in the two methods. For items satisfying the demand assumptions considered here, then, it is established quantitatively that exponential smoothing is definitely inferior to more classical methods.

II. RATIO OF Δ/σ

The mean absolute deviation of a probability distribution is given by

$$\Delta = E(|X - \mu|) .$$

For a normal distribution the ratio Δ/σ is the constant $\sqrt{2/\pi} \cong 0.8$. Brown [Ref. 1] in his work on smoothing and prediction states,

"Hence, the mean absolute deviation is proportional to the standard deviation, and the ratio may depend on the form of the distribution p(t), but dependence is slight."

Now, as indicated by Zehna in [Ref. 7], this is not always the case. For example, consider a Poisson distribution with parameter λ. Δ/σ is then a function of λ and it may be much smaller than 0.8 for small values of λ [Ref.8]. However, there are some distributions where the approximation 0.8 for the ratio Δ/σ is a good one under certain conditions. The quote above is a rather general statement which may be justified as follows:

Let X be a continuous random variable with probability density function (p.d.f.) f(x) and let $\underline{u}$ and $\underline{l}$ be the upper and lower limits, respectively, of the random variable X on the real line. Then

$$\Delta = E(|X - \mu|) = \int_{\ell}^{u} |x - \mu| \; f(x)dx$$

or

$$\Delta = -\int_{\ell}^{\mu} xf(x)\,dx + \mu \int_{\ell}^{\mu} f(x)\,dx + \int_{\mu}^{u} xf(x)\,dx - \mu \int_{\mu}^{u} f(x)\,dx.$$

Hence,

$$\Delta = 2 \int_{\ell}^{\mu} F(x)\,dx$$

Here, $F(x)$ is the cumulative distribution of X.

Then

$$\frac{\Delta}{\sigma} = \frac{2}{\sigma} \int_{\ell}^{\mu} F(x)\,dx$$

Let $x = \sigma y + \mu$. Then,

$$\frac{\Delta}{\sigma} = 2 \int_{\frac{\ell-\mu}{\sigma}}^{\mu} F(\sigma y + \mu)\,dy$$

Now suppose that $F(x)$ is any distribution function such that

$$\frac{F(x)}{\Phi(\frac{x-\mu}{\sigma})} \rightarrow 1 \quad , \quad \text{as} \quad \frac{\mu}{\sigma} \rightarrow \infty$$

where $\Phi(z)$ is the cumulative standard normal distribution function. Then for large values of μ/σ

$$F(\sigma y + \mu) \cong \Phi(y)$$

Hence

$$\frac{\Delta}{\sigma} \cong 2 \int_{\frac{\ell-\mu}{\sigma}}^{o} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} Z^2} dz\ dy$$

Integrating by parts, the ratio then becomes $\Delta/\sigma \cong \sqrt{2/\pi}$. There are some distributions for which the above approximation holds; for example erlang and chi-square. But for other distributions such a generalization may not apply. For instance, let X have the following mass function

$$p(x) = \begin{cases} 0.1 & \text{if} \quad x = 0 \\ 0.8 & \text{if} \quad x = a \\ 0.1 & \text{if} \quad x = 2a \end{cases}$$

Then

$$\Delta = 0.2a$$

$$\sigma^2 = 1.2a$$

and

$$\Delta/\sigma = .182.$$

The error made by assuming $\Delta/\sigma \cong 0.8$ is 0.618, or 77% , which is not negligible.

III. PROBABILITY DISTRIBUTION OF $\hat{\Delta}$ AND OTHER ASSOCIATED STATISTICS

Of central interest in the use of MAD is the ability to estimate the demand variability. The variance of the forecast error is always a function of the variance of demand, σ^2. Therefore the MAD of the forecast errors is also a function of σ. Since MAD is a linear function of the standard deviation [Ref. 1] and if MAD of the forecast errors is estimated, then σ can be estimated as described in [Ref. 1]. But in this estimation there are two points to be considered.

1. Invariance Principle:

σ and Δ are related to each other by a constant, i.e.,

$$\sigma = c\Delta \text{ .}$$

But this relation is in the population, and it does not mean that the estimator of σ and Δ will enjoy the same relationship necessarily. If these estimates mentioned above were MLE, then the invariance principle does hold. However, the definition of $\hat{\Delta}$ indicates that this is not the case.

2. The Constant of Proportionality of Δ/σ:

If the underlying demand distribution is not normal, then the assumption that the proportionality constant is approximately 0.8 needs to be validated. For this

case the current model can be applied by a slight modification on $\tilde{\sigma}$, that is, replacing $\sqrt{\pi/2}$ by the reciprocal of the appropriate proportionality constant.

In this section we assume that demand is Normal (a,σ^2), where a is a constant. Derivation of the probability distribution of $\hat{\Delta}$ requires probability distribution for the following statistics:

A. EXPONENTIAL SMOOTHING OF PAST OBSERVATIONS

(Estimate of the Expected Value of Demand)

$$S_t(x) = \alpha \sum_{k=0}^{t-1} \beta^k X_{t-k}$$

Since all X's are mutually independent, the distribution of $S_t(x)$ for large values of t is known to be a normal distribution with mean a and variance $\sigma^2(1 + \frac{\alpha}{1+\beta})$.

B. FORECAST ERRORS

$$e_{t-i} = X_{t-i} - \alpha \sum_{k=0}^{t-i-2} \beta^k X_{t-i-k-1} \qquad i < t$$

e_{t-1} has a normal distribution with the following mean and variance:

$\mu = 0$ for large values of t;

$$V(e_{t-i}) = \sigma^2 \left(1 + \frac{\alpha}{1+\beta}\right)$$

for large values of t.

C. JOINT DISTRIBUTION OF FORECAST ERRORS

Let U and V be t-dimensional vectors defined as

$$V = \begin{bmatrix} e_t \\ e_{t-1} \\ \vdots \\ e_2 \\ e_1 \end{bmatrix} \qquad U = \begin{bmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_2 \\ U_1 \end{bmatrix}$$

where $U_1, \ldots, U_t$ are constants. Let Y be defined as

$$Y = U^T V$$

or

$$Y = \sum_{i=1}^{t} U_i e_i$$

Substituting e_i in terms of the x's,

$$Y = \sum_{i=1}^{t} U_i \left(x_i - \alpha \sum_{k=0}^{i=2} \beta^k x_{i-k-1} \right)$$

or

$$Y = \sum_{i=1}^{t} U_i X_i - \alpha \sum_{i=1}^{t} \sum_{k=0}^{i-2} U_i \beta^k x_{i-k-1}$$

Hence,

$$Y = \sum_{i=0}^{t-1} X_{t-i} \left(U_{t-i} - \alpha \sum_{k=0}^{i-1} \beta^k U_{t-k} \right)$$

Since the coefficients of the X's are constants and the X's are mutually independent normal variables, it follows that Y has a normal distribution. Hence by the definition of a multivariate normal distribution [Ref. 2], the forecast errors have a multivariate normal distribution. The correlation coefficient between e_{t-i} and e_{t-j}, where $i < j$, can be computed as follows.

Let

$$Y = e_{t-i} + e_{t-j} \qquad \text{where} \quad i < j < t.$$

$$V(Y) = V(e_{t-i}) + (V(e_{t-j}) + 2Cov(e_{t-i}, e_{t-j}),$$

or

$$Cov(e_{t-i}, e_{t-j}) = \frac{1}{2} \left[V(Y) - V(e_{t-i}) - V(e_{t-j}) \right] \tag{3-1}$$

Rewriting in terms of the X's,

$$Y = X_{t-i} - \alpha X_{t-i-1} - \alpha\beta X_{t-i-2} - \cdots\cdots - \alpha\beta^{j-i-1} X_{t-j} -$$

$$\alpha\beta^{j-i} X_{t-j-1} - \cdots\cdots + X_{t-j} - \alpha X_{t-j-1} - \alpha\beta X_{t-j-2} - \cdots\cdots$$

Rearranging terms on the right-hand side,

$$Y = X_{t-i} - \alpha \sum_{k=0}^{j-i-2} \beta^k X_{t-i-1-k} + (1-\alpha\beta^{j-i-1})X_{t-j} -$$

$$(1+\beta^{j-i}) \; \alpha \sum_{k=0}^{t-j-1} \beta^k X_{t-j-1-k}$$

Since the terms in the above expression are mutually independent, the variance of Y for large values of t can be found as follows:

$$V(Y) = \sigma^2 \left(1 + \alpha^2 \; \frac{1-\beta^{2(j-i-1)}}{1-\beta^2} + (1-\alpha\beta^{j-i-1})^2 + \frac{\alpha^2(1+\beta^{j-i})^2}{1-\beta^2}\right)$$

Substituting V(Y), $V(e_{t-i})$, and $V(e_{t-j})$ into (3-1),

$$\mathrm{Cove}(e_{t-i}, e_{t-j}) = \frac{\sigma^2}{2} \left(1 + \frac{\alpha}{1+\beta} \left(1-\beta^{2(j-i-1)} + (1+\beta^{j-i})^2\right) + \right.$$

$$\left. (1-\alpha\beta^{j-i-1})^2 - \frac{4}{1+\beta} \right)$$

Hence, the asymptotic correlation coefficient between e_{t-i} and e_{t-j} becomes

$$\rho = \frac{1+\beta}{4} \left(1 + \frac{\alpha}{1+\beta} \left(1-\beta^{2(j-i-1)} + (1+\beta^{j-i})^2\right) + \right.$$

$$\left. (1-\alpha\beta^{j-i-1})^2 \right) - 1 \qquad (3\text{-}2)$$

Now consider the following function,

$$\rho' = \frac{1+\beta}{4}\left(1 + \frac{\alpha}{1+\beta}\,(1-\beta^{2y-2} + (1+\beta^{y})^{2}) + (1-\alpha\beta^{y-1})^{2}\right) - 1$$

Here y is a continuous variable. Then

$$\frac{d\rho'}{dy} = \frac{\alpha}{4}\left(-\frac{2\beta^{2y}}{\beta^{2}}\log\beta + 2\beta^{y}\log\beta + 2\beta^{2y}\log\beta\right) +$$

$$\frac{1+\beta}{4}\left(-\frac{2\alpha\beta^{y}}{\beta}\log\beta + \frac{2\alpha^{2}\beta^{2y}}{\beta^{2}}\log\beta\right)$$

Rearranging the terms,

$$\frac{d\rho'}{dy} = \beta^{y}\log\beta\left(-\frac{\alpha\beta^{y}}{2\beta^{2}} + \frac{\alpha}{2} + \frac{\alpha\beta^{y}}{2} - \frac{(1+\beta)\alpha}{2\beta} + \frac{\alpha^{2}(1+\beta)\beta^{y}}{2\beta^{2}}\right)$$

or

$$\frac{d\rho'}{dy} = \frac{\beta^{y}\log\beta}{2}\left(\alpha - \frac{(1+\beta)\alpha}{\beta}\right) = -\frac{\alpha\beta^{y}\log\beta}{2\beta}$$

Now $\log\beta$ is negative; therefore, the first derivative of ρ', with respect to y is positive. Hence ρ', is a monotonic increasing function of y. Then positive integral values of y also lie on this same continuous curve. Actually, ρ coincides with ρ', whenever y is a positive integer. Hence ρ is a monotonic increasing function of j-i. The supremum of ρ is found by letting $j-i \to\infty$ in (3-2):

$$\begin{aligned}\text{Sup }\rho &= \frac{1+\beta}{4}\left(1 + \frac{\alpha}{1+\beta}\,(1+1) + 1\right) - 1\\ &= 0\end{aligned}$$

The minimum value of ρ is found by letting j-i = 1:

$$\rho_{min} = \frac{1+\beta}{4}\left(1+ \frac{\alpha}{1+\beta}\left((1-1) + (1+\beta)^2\right) + (1-\alpha)^2\right) - 1$$

$$= \frac{1+\beta}{4}\left(1 + (1+\beta)\alpha + \beta^2\right) - 1$$

$$= \frac{1+\beta}{2} - 1$$

The minimum value of β may be zero; hence, $\rho_{min} = -0.5$.

D. DISTRIBUTION OF $|e_{t-i}|$

Rewriting $|e_{t-i}|$ as

$$|e_{t-i}| = + \sqrt{e^2_{t-i}}$$

The term

$$\frac{e^2_{t-i}}{\sigma^2(1+ \frac{\alpha}{1+\beta})}$$

has, asymptotically, the chi-square distribution with one degree of freedom [Ref. 3].

The term

$$+ \sqrt{\frac{e^2_{t-i}}{\sigma^2(1+ \frac{\alpha}{1+\beta})}}$$

has, asymptotically, the chi distribution. Hence the mean value and variance of e_{t-i} is given by

$$E(|e_{t-i}|) = \sigma\sqrt{\frac{2}{2-\alpha}}\sqrt{\frac{2}{\pi}} \tag{3-3}$$

$$V(|e_{t-i}|) = \frac{\pi-2}{\pi}\sigma^2\left(1 + \frac{\alpha}{1+\beta}\right) \tag{3-4}$$

An attempt to derive the probability distribution of Δ was made utilizing the change-of-variable technique, and by defining the following transformation:

$$\Delta = \alpha \sum_{k=0}^{t-1} \beta^k |e_{t-k}|$$

$$Y_1 = e_{t-1}$$

$$\vdots$$

$$Y_{t-1} = e_1$$

The inverse transformation becomes

$$e_1 = \pm Y_{t-1}$$

$$e_2 = \pm Y_{t-2}$$

$$\vdots$$

$$e_t = \pm\left(\Delta - \sum_{k=1}^{t-1} \beta^k (\pm Y_k)\right)$$

This transformation is obviously not one-to-one; there are 2^t sets in the domain of the transformation that are

transformed into a single set in the image. For all cases the absolute value of the Jacobian of transformation is one. The joint density of $\hat{\Delta}$, $Y_1, \ldots, Y_{t-1}$ then becomes

$$f(y_1, \ldots . y_{t-1}, \delta) = \sum_{i=1}^{2^t} g\left(\pm y_1, \pm y_2, \ldots \pm (\delta - \alpha \sum_{k=1}^{t-1} \beta^k (\pm y_k)\right)$$

where $g(x_1, x_2, \ldots, x_n)$ is the p.d.f. of a multivariate normal distribution, and for every i one combination of the signs is valid. The distribution of $\hat{\Delta}$ then naturally requires the (t-1)-fold integration of the joint density of $\hat{\Delta}$, $Y_1, \ldots, Y_{t-1}$. Although every parameter in the joint density is known, the labor required to arrive at an explicitly-defined probability distribution makes the task very difficult. Therefore, the analysis concerning the dispersion of $\tilde{\sigma}$ was not completed because of the intractibility.

IV. MOMENTS OF $\hat{\Delta}$

The next step of the analysis, then, was a simulation of the inventory system in order to gain an insight concerning the distribution of $\hat{\Delta}$. It was seen that the reorder levels were fluctuating around the theoretical reorder levels, even after 1000 observations. This was related to the asymptotically finite variance of $\hat{\Delta}$ as indicated by [Ref.7]. Although computer simulations implied variance of $\hat{\Delta}$ does not vanish, the moments of $\hat{\Delta}$ were derived in order that an analysis of the simulation results might be made. The moments of $\hat{\Delta}$ were derived for large values of t and are, therefore, asymptotic.

A. MEAN VALUE OF $\hat{\Delta}$ FOR LARGE VALUES OF t.

$$\hat{\Delta} = \alpha \sum_{k=0}^{t-1} |e_{t-k}| \beta^k \qquad\qquad e_o \equiv 0$$

$$E(\hat{\Delta}) = \alpha \sum_{k=0}^{t-1} \beta^k E(|e_{t-k}|)$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{2-\alpha}} \; \sigma$$

B. VARIANCE OF $\hat{\Delta}$

$$V(\hat{\Delta}) = \alpha^2 \sum_{k=0}^{t-1} \beta^{2k} V(|e_{t-k}|) + 2\alpha^2 \sum_{i=0}^{t-2} \sum_{j=i+1}^{t-1} \beta^{i+j} \mathrm{Cov}(|e_{t-i}|, |e_{t-j}|) \tag{4-1}$$

$$\mathrm{Cov}(|e_{t-i}|, |e_{t-j}|) = E(|e_{t-i}e_{t-j}|) - E(|e_{t-i}|)E(|e_{t-j}|) \tag{4-2}$$

Thus, substituting (3-3), (3-4), and (4-2) into (4-1),

$$V(\hat{\Delta}) = \frac{\pi-2}{\pi}\sigma^2 \frac{2}{2-\alpha}\frac{\alpha}{1+\beta} - \frac{8\beta}{(1+\beta)^2\pi}\sigma^2 + 2\alpha^2 \sum_{i=0}^{t-2}\sum_{j=i+1}^{t-1} \beta^{i+j} E(|e_{t-i}e_{t-j}|) \tag{4-3}$$

This last expression for $V(\hat{\Delta})$ requires the computation of $E(|e_{t-i}e_{t-j}|)$. The following derivations concern $E(|e_{t-i}e_{t-j}|)$.

The joint distribution of e_{t-i} and e_{t-j}, as indicated in Section III, is bivariate normal. The parameters of this distribution are

$$\mu_1 = E(e_{t-i}) = 0 \qquad \sigma_1 = \sigma\sqrt{1 + \frac{\alpha}{1+\beta}}$$

$$\mu_2 = E(e_{t-j}) = 0 \qquad \sigma_2 = \sigma\sqrt{1 + \frac{\alpha}{1+\beta}}$$

and asymptotic ρ is given by (3-2). For notational convenience, let

$$X = e_{t-i}$$

$$Y = e_{t-j}$$

Then the joint p.d.f. is given [Ref. 4] by

$$f(x,y) = \frac{1}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}\, e^{-\frac{q}{2}} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

where

$$q = \frac{1}{1-\rho^2}\left(\frac{x^2}{\sigma_x^{\ 2}} - 2\rho \frac{xy}{\sigma_x \sigma_y} + \frac{y^2}{\sigma_y^{\ 2}} \right)$$

In that case

$$E(|XY|) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy|\ f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} |x| \left(-\int_{-\infty}^{0} y\ f(x,y)\,dy + \int_{0}^{\infty} y\ f(x,y)\,dy \right) dx$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{0} xy\ f(x,y)\,dydx - \int_{-\infty}^{0} \int_{0}^{\infty} xy\ f(x,y)\,dydx -$$

$$\int_{0}^{\infty} \int_{-\infty}^{0} xy\ f(x,y)\,dydx + \int_{0}^{\infty} \int_{0}^{\infty} xy\ f(x,y)\,dy\,dx$$

However, by the definition of correlation coefficient,

$$E(XY) = \int_{-\infty}^{0} \int_{-\infty}^{0} xy\ f(x,y)\,dydx + \int_{-\infty}^{0} \int_{0}^{\infty} xy\ f(x,y)\,dy\,dx +$$

$$\int_{0}^{\infty} \int_{-\infty}^{0} xy\ f(x,y)\,dydx + \int_{0}^{\infty} \int_{0}^{\infty} x\ y\ f(x,y)\,dy\,dx$$

$$= \sigma_x \sigma_y\, \rho$$

Thus,

$$E(|XY|)+E(XY) = 2\left(\int_{-\infty}^{0} \int_{-\infty}^{0} xy \; f(x,y)\,dydx + \int_{0}^{\infty}\int_{0}^{\infty} xy\, f(x,y)\,dydx \right)$$

Making a change of variable in the first integral, let

$$z = -x$$

$$u = -y$$

then

$$\int_{-\infty}^{0} \int_{-\infty}^{0} xy\, f(x,y)\,dydx = \int_{0}^{\infty} \int_{0}^{\infty} zu \; f(-z,-u)\,dudz$$

Since $f(x,y) = f(-x,-y)$, it follows that

$$\int_{-\infty}^{0} \int_{-\infty}^{0} xyf(x,y)\,dydx = \int_{0}^{\infty} \int_{0}^{\infty} zu \; f(z, u)\,du\,dz$$

Thus

$$E(|XY|) + E(XY) = 4 \int_{0}^{\infty} \int_{0}^{\infty} xy \; f(x,y)\,dydx \tag{4-4}$$

The right-hand side of (4-4), without the coefficient 4, is

$$\int_{0}^{\infty} \int_{0}^{\infty} xy\, f(x,y)\,dydx = \int_{0}^{\infty} \frac{x e^{-\frac{x^2}{2\sigma_x^{2}}}}{\sigma_x\sqrt{2\pi}} \int_{0}^{\infty} \frac{ye^{-\frac{(y-b)^2}{2\sigma_x^{2}(1-\rho^2)}}}{\sqrt{2\pi}\;\sigma_y\sqrt{1-\rho^2}}\,dy\,dx \tag{4-5}$$

where b is the conditional mean of Y given X, and is given by [Ref. 4],

$$b = \frac{\sigma_y}{\sigma_x}\;\rho x$$

The conditional standard deviation of Y given X is given by

$$c = \sigma_y \sqrt{1-\rho^2}$$

Let

$$\frac{y - b}{c} = z \text{ , then}$$

$$\int_0^\infty \frac{x e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} \int_0^\infty \frac{y e^{-\frac{(y-b)^2}{2c^2}}}{c\sqrt{2\pi}} dydx = \int_0^\infty \frac{x e^{-\frac{x^2}{\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} \int_{-\frac{b}{c}}^\infty \frac{(cz+b) e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz\, dx$$

(4-6)

Integrating the inner term first:

$$\int_{-\frac{b}{c}}^\infty \frac{cz+b}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{c}{\sqrt{2\pi}} e^{-\frac{b^2}{2c^2}} + b\,(1 - \Phi(-\frac{b}{c}))$$

where $\Phi(x)$ is the cumulative normal distribution. Substituting this expression into (4-6),

$$\int_0^\infty \frac{x e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} \int_0^\infty \frac{y e^{-\frac{(y-b^2)^2}{2c^2}}}{c\sqrt{2\pi}} dydx = \int_0^\infty \frac{cx\ e^{-\frac{x^2}{2\sigma_x^2} - \frac{b^2}{2c^2}}}{\sigma\ x\ 2\pi} dx +$$

$$\int_0^\infty \frac{bx\ e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} dx - \int_0^\infty \frac{b\ \Phi(-\frac{b}{c})e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} dx \qquad (4\text{-}7)$$

Denote the terms on the right-hand side as 1, 2, and 3, respectively, and compute each one separately as follows:

1. Substituting values of c and b into 1,

$$\int_0^\infty \frac{cx\, e^{-\frac{x^2}{2\sigma_x} - \frac{\rho^2 x^2}{2(1-\rho^2)\sigma_x^2}}}{2\pi\sigma_x}\, dx = \int_0^\infty \frac{\sigma_y\sqrt{1-\rho^2}\, x\, e^{-\frac{x^2}{2\sigma_x^2(1-\rho^2)}}}{2\pi\sigma_x}\, dx$$

$$= \frac{\sigma_y\sqrt{1-\rho^2}}{2\pi}\, \sigma_x(1-\rho^2)$$

$$= \frac{\sigma_y\sigma_x(1-\rho^2)^{\frac{3}{2}}}{2\pi}$$

2. Substituting the value of b into 2,

$$\int_0^\infty \frac{\rho\sigma_y x^2\, e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x^2\sqrt{2\pi}}\, dx = \frac{\sigma_y\rho}{\sigma_x^2\sqrt{2\pi}} \int_0^\infty x^2\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx$$

$$= \frac{\sigma_y\sigma_x\rho}{2}$$

3. Substituting the values of c and b into 3,

$$\int_0^\infty \Phi\left(\frac{-\rho x}{\sigma_x\sqrt{1-\rho^2}}\right) \frac{x\, e^{-\frac{x^2}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}} \quad \frac{\sigma_y}{\sigma_x} \times \rho =$$

$$\frac{\sigma_y\rho}{\sigma_x^2\sqrt{2\pi}} \int_0^\infty x^2\, e^{-\frac{x^2}{2\sigma_x^2}} \int_{-\infty}^{-p} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt\, dx$$

where $$p = \frac{\rho x}{\sigma_x\sqrt{1-\rho^2}} \tag{4-8}$$

Let $x/\sigma_x = y$ in (4-8); then (4-8) becomes

$$\frac{\sigma_x \sigma_y \rho}{\sqrt{2\pi}} \int_o^\infty y^2 e^{-y^2/2} \int_{-\infty}^{-r} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt\ dy$$

where $r = \dfrac{\rho y}{\sqrt{1-\rho^2}}$

Integrating by parts,

$$du = y\ e^{-\frac{y^2}{2}} dy \qquad v = y \int_{-\infty}^{-r} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$$

$$u = -\ e^{-\frac{y^2}{2}} \qquad dv = \left(\int_{-\infty}^{-r} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt - y\ \frac{e^{-\frac{r^2}{2}}}{} \frac{\rho}{\sqrt{1-\rho^2}} \right) dy$$

Eq. (4-8) becomes

$$K \left| -y\ e^{-\frac{y^2}{2}} \int_o^{-r} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \right|_o^\infty - K \int_o^\infty \frac{\rho y\ e^{-\frac{y^2}{2\sqrt{1-\rho^2}}}}{\sqrt{1-\rho^2}\ \sqrt{2\pi}} dy\ +$$

$$K \int_o^\infty e^{-\frac{y^2}{2}} \int_{-\infty}^{-r} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt\ dy\ .$$

where

$$K = \frac{\sigma_x \sigma_y \rho}{\sqrt{2\pi}}$$

The first term in the above equation is zero; hence, (4-8) becomes

$$-\frac{\sigma_x\sigma_y\rho^2(1-\rho^2)^{\frac{1}{2}}}{2\pi} + K\int_0^\infty e^{-\frac{y^2}{2}}\int_{-\infty}^{-r}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt\,dy \tag{4-9}$$

Now considering the second term in (4-9),

$$\int_0^\infty e^{-\frac{y^2}{2}}\int_{-\infty}^{-r}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt\,dy = \int_0^\infty e^{-\frac{y^2}{2}}\left(0.5+\int_0^{-r}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt\right)dy \tag{4-10}$$

This result, due to the fact that y changes over positive numbers and ρ is between -.5 and zero, makes the argument of $\Phi(\cdot)$ positive. Hence (4-10) becomes

$$\frac{1}{2}\sqrt{\frac{\pi}{2}} + \int_0^\infty e^{-\frac{y^2}{2}}\int_0^{-r} e^{-\frac{t^2}{2}}\,dt\,dy = M$$

Let

$$t = v\sqrt{2}$$

$$y = \sqrt{2u}$$

then

$$M = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \int_0^\infty \frac{e^{-u}}{\sqrt{2u}}\int_0^{-s} e^{-v^2}\sqrt{2}\,dv\,du$$

where $\quad s = \dfrac{\rho\sqrt{u}}{\sqrt{1-\rho^2}}$

$$M = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-u}}{u^{\frac{1}{2}}} \int_0^{-s} \sum_{n=0}^{\infty} (-1)^n \frac{v^{2n}}{n!} \, dv \, du$$

Since the series is uniformly convergent, it follows that

$$M = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-u}}{u^{\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{(-\rho)^{2n+1}}{(\sqrt{1-\rho^2})^{2n+1}} \frac{u^{n+\frac{1}{2}}}{n!(2n+1)} \, du$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(-\rho)^{2n+1}}{(\sqrt{1-\rho^2})^{2n+1}} \frac{1}{2n+1} \frac{\int_0^\infty u^n e^{-u} \, du}{n!}$$

or,

$$M = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(-\frac{\rho}{\sqrt{1-\rho^2}}\right)^{2n+1}}{2n+1}$$

It was previously shown that ρ is always between -.5 and 0. Therefore,

$$-1 < -\frac{\rho}{\sqrt{1-\rho^2}} < 1$$

Hence the series in the above expression is the expansion of Arctan $\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$ [Ref. 6] .

Thus Eq. (4-10) becomes

$$\frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \text{ Arctan}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right) \tag{4-11}$$

Substituting (4-11) into (4-9) yields

$$-\frac{\sigma_x\sigma_y\rho^2(1-\rho^2)^{\frac{1}{2}}}{2\pi} + \frac{\sigma_x\sigma_y\rho}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \text{ Arctan }\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right)$$

Combining the results for 1, 2, and 3 into (4-7), and substituting into (4-5) yields

$$\int_0^\infty \int_0^\infty xy\ f(x,y)\,dy\,dx = \frac{\sigma_y\sigma_x(1-\rho^2)^{\frac{3}{2}}}{2\pi} + \frac{\sigma_x\sigma_y\rho}{2} +$$

$$\frac{\sigma_x\sigma_y\rho^2(1-\rho^2)^{\frac{1}{2}}}{2\pi} - \frac{\sigma_x\sigma_y\rho}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{2\pi}} \text{ Arctan}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right)$$

Thus,

$$E(|XY|) = 4\left(\frac{\sigma_y\sigma_x}{2\pi}(1-\rho^2)^{\frac{1}{2}} + \frac{\sigma_x\sigma_y\rho}{4} - \frac{\sigma_x\sigma_y}{2\pi} \text{ Arctan }\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right) -$$

$$\sigma_x\sigma_y\rho$$

$$= \frac{2\sigma_x\sigma_y}{\pi}\left((1-\rho^2)^{\frac{1}{2}} - \rho \text{ Arctan }\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right) \tag{4-12}$$

But σ_x and σ_y are both equal to $\sigma\sqrt{1+\frac{\alpha}{1+\beta}}$. Hence,

$$E(|XY|) = \frac{2\sigma^2(1+\frac{\alpha}{1+\beta})}{\pi}\left((1-\rho^2)^{\frac{1}{2}} - \rho \text{ Arctan } (\frac{-\rho}{\sqrt{1-\rho^2}})\right) \quad (4\text{-}13)$$

Therefore,

$$V(\hat{\Delta}) = \frac{\pi-2}{\pi}\sigma^2 \frac{2\alpha}{(1+\beta^2)} - \frac{8\beta}{\pi(1+\beta)^2}\sigma^2 + 2\alpha^2 \sum_{i=0}^{t-2}\sum_{j=i+1}^{t-1} \beta^{i+j} E(|e_{t-i}e_{t-j}|) \quad (4\text{-}14)$$

where $E(|e_{t-j}e_{t-i}|)$ is given by (4-13).

It is clear that to arrive at an expression for $V(\hat{\Delta})$ in the form of (4-14) many tedious and difficult calculations must be made. In fact, in order to arrive at an explicit value for $V(\hat{\Delta})$, one must deal with an arctangent term. However, because σ^2 is common to each term, the ratio $V(\hat{\Delta})/\sigma^2$ can be defined in terms of the known constants, and an approximate numerical answer can be found for the ratio $V(\hat{\Delta})/\sigma^2$. The computer program in Appendix B computes the first two terms in (4-13) exactly, and the third term approximately. Briefly, the computation of the third term was accomplished as follows:

Eq. (4-13) is a bounded number for a given α and β. It cannot be made greater than some finite number d, whatever be the value of ρ . (Note that ρ is the only variable in (4-13) that depends on i and j, and it is bounded.)

Therefore, in the double summation of (4-14), for some finite value of i and j, higher order terms will get smaller than a specified number m which, in this case, was chosen as 10^{-10}. The program was written so that the higher terms smaller than 10^{-10} were not included in the result, and were subsequently dropped from the summation. The upper bound of the error introduced by omitting the higher terms from the summation can be found as follows:

$E(|XY|)$ is bounded from above by

$$d = \frac{4}{\pi(1+\beta)} \quad \frac{\pi}{2} = \frac{2}{1+\beta}$$

Since the summation is infinite,

$$\text{Upper Bound of Error} = 2\alpha^2 \frac{2}{1+\beta} 10^{-10} \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \beta^{i+j}$$

$$= \frac{4\alpha^2 \, 10^{-10} \, \beta}{(1-\beta^2)^2}$$

$$= \frac{4\beta}{(1+\beta)^2} 10^{-10}$$

First, this error is additive and hence is added to the result. Second, whatever be the value of β, it is always much smaller than one. Hence, the results contained in Table 1, obtained by the computer program for various values of α were almost exact.

SMOOTHING CONSTANT	VARIANCE OF $(\hat{\Delta})/\sigma^2$	STANDARD DEVIATION OF $(\tilde{\sigma})/\sigma$
0.10	0.0204	0.1745
0.15	0.0325	0.2173
0.20	0.0461	0.2553
0.25	0.0614	0.2905
0.30	0.0785	0.3239
0.35	0.0979	0.3561
0.40	0.1196	0.3876
0.45	0.1440	0.4187
0.50	0.1716	0.4495
0.55	0.2026	0.4802
0.80	0.4267	0.6341

Table 1. Variance of $\frac{\hat{\Delta}}{\sigma^2}$ and Standard Deviation of $\frac{\tilde{\sigma}}{\sigma}$.

V. COMPARISON OF EXPONENTIAL SMOOTHING AND MLE BY COMPUTER SIMULATION

For the reason explained previously, simulation was utilized to explore the parts of the model where theoretical work was very difficult, and was also used to compare the exponential smoothing with MLE. The reason that MLE was chosen as the alternative procedure was because it has the property of an asymptotically vanishing variance. As indicated in [Ref. 8], the maximum likelihood estimator, $\hat{\sigma}$, of σ is given by the formula

$$\hat{\sigma} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}$$

where $\bar{x}$ is the sample mean. The statistic $\hat{\sigma}$ has a chi-square distribution [Ref. 3]; in fact,

$$E(\hat{\sigma}) = \sqrt{1 - \frac{1}{n}} \sqrt{\frac{2}{n-1}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \sigma$$

and

$$V(\hat{\sigma}) = \frac{n-1}{n} \sigma^2 \left(1 - \frac{2}{n-1} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \right)^2 \right)$$

From these formulas,

$$\lim_{n \to \infty} E(\hat{\sigma}) = \sigma$$

$$\lim_{n \to \infty} V(\hat{\sigma}) = 0$$

The computer program used for the simulation is included in Appendix A. The simulation procedure and ensuing results are as follows:

One thousand normally-distributed random numbers, representing the demand, were generated by a pseudorandom number generator. Various values of the parameters, μ, σ, and ψ were used. For every triplet of the parameters, reorder levels, risk levels attained, estimates of σ and sample mean, and sample standard deviation of $\hat{\sigma}$ and of the estimate of σ by MAD were computed. The value of α was 0.2 for all cases. Table 2 presents the results achieved by the simulation. As Table 2 indicates, the theoretical values of the standard deviations of both estimates of σ were in complete agreement with the sample standard deviations of these same estimates. The variance of $\tilde{\sigma}$ is higher than that of $\hat{\sigma}$ in every case. The last two columns give the mean square error (MSE) of $\tilde{\sigma}$ and $\hat{\sigma}$. The MSE is given by the variance plus the square of the bias. Although $\hat{\sigma}$ is biased for 1000 observations, its bias can be neglected. Then the MSE of $\tilde{\sigma}$ and $\hat{\sigma}$ are determined by their variances. The comparison of the MSE columns are quite striking; the MSE for $\tilde{\sigma}$ reaches values as high as 100 times that for $\hat{\sigma}$. By noting the similarity between the theoretical values and the simulation results, the effect on reorder levels was analyzed as follows:

Computation of reorder levels was made by utilizing Eq. (3-3). First, the theoretical values of the reorder

PARAMETER PAIRS (μ, σ)	AVERAGE $\tilde{\sigma}$	AVERAGE $\hat{\sigma}$	BIAS $\tilde{\sigma}$	BIAS $\hat{\sigma}$	SAMPLE SD OF $\tilde{\sigma}$	THEORETICAL SD OF $\tilde{\sigma}$	SAMPLE SD OF $\hat{\sigma}$	THEORETICAL SD OF $\hat{\sigma}$	MSE $\tilde{\sigma}$	MSE $\hat{\sigma}$
10,1	.99	.96	-.01	-.04	.26	.255	.05	.02	.065	.0004
10,	1.01	.99	.01	-.01	.26	.225	.05	.02	.065	.0004
20,2	1.95	1.93	-.05	-.07	.50	.510	.07	.04	.260	.0016
30,3	2.87	2.78	-.13	-.22	.75	.765	.20	.06	.585	.0036
40,4	4.28	4.22	.28	.22	1.05	1.020	.13	.03	1.040	.0081
50,5	5.06	5.09	.06	.09	1.36	1.275	.26	.11	1.625	.0121
100,10	10.35	9.99	.85	-.01	2.63	2.553	.43	.22	6.517	.0484
200,20	20.06	12.77	.06	-.23	5.81	5.100	.87	.44	26.01	.1236
300,30	30.28	29.33	.28	-.67	7.36	7.650	1.51	.67	58.52	.4489
400,40	41.23	39.98	1.23	-.02	10.40	10.200	1.61	.89	104.04	.7921
500,50	46.91	46.52	-3.09	-3.48	11.85	12.750	2.42	1.11	162.56	1.232
600,60	60.20	57.63	.20	-2.37	15.43	15.300	4.43	1.34	234.09	1.795
700,70	67.42	68.01	-1.58	-1.99	18.71	17.850	5.01	1.56	318.62	2.433
800,80	77.57	75.27	-2.43	-4.63	19.44	20.400	4.54	1.98	416.16	3.920
900,90	91.28	87.48	1.28	-2.52	22.23	22.950	4.85	2.01	526.70	4.040
1000,100	99.20	95.77	-.80	-4.23	25.10	25.530	4.03	2.23	651.78	4.272

Table 2. Sample Characteristics

levels were computed for every fixed parameter triplet μ, σ, ψ . Then the reorder levels, found by smoothing and MLE, were compared with the theoretical values. This comparison is summarized in Table 3. The value Q_1 in Table 4 is the percentage (rounded off to the nearest integer) of 1000 reorder levels found to be within one unit of the theoretical reorder level. The first column under each of the headings Q_1, P_{50}, and P_{95} lists the results for smoothed estimates; the second column under each heading lists MLE. The quantity P_{50} represents the number of units about the theoretical reorder level within which 50 per cent of the computed reorder levels were found; P_{95} is a similar computation for 95 per cent of the reorder levels. Table 3 clearly illustrates the superiority of MLE compared to exponential smoothing. For the case where smoothing was used to estimate σ, reorder levels fluctuated around the theoretical reorder level, causing either an excessive purchase, or imposing more risk than desired. The magnitudes of these fluctuations are clearly evident from the table .

Table 4 indicates the risk levels attained by both methods at the 1000th observation. For every predetermined parameter pair μ and σ, five different risk levels were selected. For every triplet, the theoretical reorder levels were compared with the computed reorder levels of both procedures. The actual risk levels attained were then computed.

PARAMETER TRIPLETS	Q_1		P_{50}		P_{95}	
40,4,.01	27	95	2	0	6	0
40,4,.11	41	94	2	0	4	0
40,4,.50	49	92	1	0	3	1
50,5,.01	22	96	3	0	8	0
50,5,.11	32	98	2	0	5	0
50,5,.50	46	99	1	0	3	0
100.10,.01	12	90	5	0	14	1
100,10,.11	18	94	4	0	11	0
100,10,.50	26	94	2	0	6	0
200,20,.01	5	71	10	1	29	6
200,20,.11	9	82	6	1	18	3
200,20,.50	10	84	5	1	14	2
300,30,.01	4	81	12	1	37	6
300,30,.11	7	74	8	1	24	3
300,30,.50	8	51	7	1	18	3
400,40,.01	2	15	20	3	53	10
400,40,.11	4	32	13	2	37	6
400,40,.50	5	75	10	1	27	3
500,50,.01	3	1	24	9	58	11
500,50,.11	3	4	15	4	38	6
500,50,.50	4	73	11	1	29	4
600,60,.01	1	12	27	3	77	17
600,60,.11	3	10	18	3	51	5
600,60,.50	3	1	14	4	42	12
700,70,.01	1	33	32	3	93	16
700,70,.11	3	35	20	2	63	10
700,70,.50	4	16	15	2	45	15
800,80,.01	1	0	37	12	98	27
800,80,.11	2	0	25	8	64	18
800,80,.50	3	32	18	2	54	9

Table 3. Percentiles for Reorder Levels.

PARAMETER TRIPLETS	Q_1		P_{50}		P_{95}	
900,90,.01	1	30	42	2	125	12
900,90,.11	2	37	29	2	83	12
1000,100,.01	1	25	45	3	126	12
1000,100,.11	1	8	29	4	87	7
1000,100,.50	1	6	25	2	68	13

Table 3.(Continued) - Percentiles for Reorder Levels.

As Table 4 indicates, the risk levels attained by MLE are found to be very close to the desired risk levels. However, the risk levels of the smoothing method were found to fluctuate over a wide range.

These results, gained by the simulation, indicated that the MLE technique is to be preferred over the technique of exponential smoothing. However, due to the lack of theoretical analysis concerning the distribution of these results are not considered conclusive. However, this analysis does indicate that whenever a constant mean demand can be justified, the MLE technique is to be preferred to the exponential smoothing technique.

(μ, σ)	.01		.05		.11		.25		.50	
(10.1)	.044	.013	.119	.061	.136	.126	.333	.276	.537	.529
(20,2)	.092	.011	.235	.053	.364	.113	.556	.251	.772	.427
(30,3)	.058	.012	.156	.055	.262	.116	.415	.253	.635	.494
(40,4)	.002	.007	.018	.041	.050	.097	.145	.237	.369	.502
(50,5)	.000	.009	.002	.046	.014	.103	.086	.240	.371	.493
(110.10)	.000	.009	.003	.048	.020	.108	.125	.249	.489	.507
(200,20)	.005	.009	.026	.046	.061	.103	.152	.238	.350	.489
(300,30)	.003	.010	.028	.049	.078	.108	.220	.246	.511	.495
(400,40)	.021	.008	.057	.044	.096	.101	.173	.238	.312	.492
(500,50)	.005	.015	.032	.063	.075	.127	.189	.264	.422	.499
(600,60)	.051	.009	.165	.048	.287	.106	.490	.244	.783	.497
(700,70)	.015	.011	.070	.052	.150	.111	.322	.247	.598	.490
(800,80)	.068	.012	.157	.055	.239	.116	.374	.253	.563	.498
(900,90)	.028	.010	.080	.048	.137	.107	.248	.245	.430	.497
(1000,100)	.001	.009	.014	.046	.045	.102	.148	.232	.405	.475

Table 4. Risks Attained by Both Procedures for Various Risk Levels.

Note: The first column under each heading lists the risk for the exponential smoothing procedure.

APPENDIX A

SIMULATION

```
C       IN THE FOLLOWING PROGRAM TWO METHODS OF ESTIMATION OF
C       MEAN AND STANDARD DEVIATION OF A STATIONARY PROCESS
C       ARE CONTRASTED BY USING SIMULATION TECHNIQUES
C       METHODS ARE:
C       1.THE EXPONENTIALLY SMOOTHED DATA AS THE ESTIMATE OF
C       MEAN AND THE EXPONENTIALLY SMOOTHED ABSOLUTE VALUES
C       OF ERRORS AS THE ESTIMATE OF STANDARD DEVIATION TIMES
C       A KNOWN CONSTANT
C       2.MAXIMUM LIKELIHOOD ESTIMATES
C       ALSO IN THE PROGRAM THE EFFECTS OF BOTH ESTIMATION
C       PROCEDURES ON THE RISK AND REORDER LEVELS ARE SIMULA-
C       TED BY SAMPLE SIZE 1000

        DIMENSION PK(6),T(6),XL(6),XP(6),XM(6)XN(6),XK(6)
       1X1(800,5),X2(1000,5),G1(1005),G2(1005),Z(10),TX(6)
       2TY(6),RISKX(6),RISKY(6)

C       RISK LEVELS

        PK(1)=0.01
        PK(2)=0.05
        PK(3)=0.11
        PK(4)=0.25
        PK(5)=0.50
        READ(5,131) (T(I),I=1,5)
  131   FORMAT(5F4.2)
        S=10.0
        XJ=0.0
        Y=URN(0)
        DO 821 MT=1,16
        XJ=XJ+1.0
        DO 1 J=1,5
        DO 1 I=1,800
        X1(I,J)=0.0
    1   X2(I,J)=0.0
        ZL=S

C       SMOOTHING CONSTANT ALPHA IS 0.2

        BETA=0.8
        WRITE(6,879) S,STDEX
  879   FORMAT('1'T45,'MEAN=',F6.1,10X,'S.DEV=',F6.1)
        DELTA=0.798*SQRT(2.0/(1.0+BETA))*STDEX

C       COMPUTATION OF THE THEORETICAL REORDER LEVELS

        DO 11 I=1,5
   11   Z(I)= ZL+STDEX*T(I)
        WRITE(6,150) (Z(I),I=1,5)
  150   FORMAT(//T10,'THEORETICAL REORDER LEVELS',5F10.3//)
        V=1.0
        SUM=ZL
        TOT=0.0
        B1=0.0
        B2=0.0
        B3=0.0
        B4=0.0

C       GENERATION OF NORMALLY DISTIBUTED NUMBERS WHICH HAVE
C       MEAN S AND STANDARD DEVIATION STDEX
```

```
      DO 21 I=1,1000
      CALL NORMAL (0.0,STDEX,X)
      Y=ZL+X

C     METHOD 1

      DELTA=0.2*ABS(S-Y)+0.8*DELTA
      S=0.2*Y+0.9*S
      SDEV=1.188*DELTA
      V=V+1.0
      SUM=SUM+Y

C     METHOD 2

      XBAR=SUM/V
      TOT=TOT+(XBAR-Y)**2
      STDEV=SQRT(TOT/(V-1.0))
      G1(I)=SDEV
      G2(I)=STDEV
      DO 20 J=1,5
      XL(J)=S+SDEV*T(J)
      XP(J)=XBAR+STDEV*T(J)
      XK(J)=XL(J)-Z(J)
      TX(J)=(XL(J)-ZL)/STDEX

C     RISKS ATTAINED BY BOTH METHODS

      RISKX(J)=0.5-0.5*ERF(TX(J)*0.707)
      TY(J)=(XP(J)-ZL)/STDEX
      RISKY(J)=0.5-0.5*ERF(TY(J)*0.707)
      XM(J)=XP(J)-Z(J)
      N1=XK(J)
      N2=XM(J)
      IF(N1.LT.0) N1=-N1
      IF(N2.LT.0) N2=-N2
      X1(N1+1,J)=X1(N1+1,J)+1.0
   20 X2(N2+1,J)=X2(N2+1,J)+1.0
      NS=I
      IF(NS.LE.899) GO TO 21
      WRITE(6,101) (XL(M),M=1,5),(XP(M),M=1,5)
  101 FORMAT(/T13,5F7.1,10X,5F7.1)
      WRITE(6,147) (RISKX(M),M=1,5),(RISKY(M),M=1,5)
  147 FORMAT(/T2,'ACTUAL RISK',5F7.3,10X,5F7.3)
   21 CONTINUE

C     COMPUTATION OF THE OBSERVED FREQUENCIES WITHIN THE I
C     NEIGHBORHOOD OF ACTUAL REORDER LEVEL WHERE I=1,2,3,..

      DO 26 J=1,5
      WRITE(6,105) PK(J)
  105 FORMAT(//T45,'RISK=',F5.2)
      A1=0.0
      A2=0.0
      I=1
   25 IF(A1.GE.1.0) GO TO 24
      A1=X1(I,J)/1000.0+A1
   24 IF(A2.GE.1.0) GO TO 35
      A2=X2(I,J)/1000.0+A2
   35 WRITE(6,112) I,A1,A2
  112 FORMAT(T10,'PERCENTILE BETWEEN THEO. REORDER LEVEL
     1 +AND-',I4,2F10.4)
      I=I+1
      XA=A1+A2
```

```
      IF(XA.LE.1.996) GOTO 25
   26 CONTINUE

C     SAMPLE MEAN AND STANDARD DEVIATION OF BOTH ESTIMATES
C     OF POPULATION STANDARD DEVIATION

      DO 27 I=1,1000
      B1=G1(I)+B1
   27 B2=B2+G2(I)
      B1=B1/1000.0
      B2=B2/1000.0
      DO 28 I=1,1000
      B3=B3+(G1(I)-B1)**2
   28 B4=B4+(B2-G2(I))**2
      B3=SQRT(B3/1000.0)
      B4=SQRT(B4/1000.0)
      WRITE(6,200)
  200 FORMAT(//T45,'SAMPLE MEAN AND S. DEV OF ESTIMATED S.
     1DEV')
      WRITE(6,201) B1,B3,B2,B4
  201 FORMAT(//T45,'(MAD)',5X,'MEAN=',F7.2,'SDEV=',F7.2,15X,
     1'(S2)',5X,'MEAN=',F7.2,'SDEV=',F7.2)
      S=100.0
      IF(XJ-5.0) 721,721,722
  721 S=XJ*10.0
      GO TO 821
  722 S=S*(XJ-5.0)
  821 CONTINUE
      STOP
      END

      SUBROUTINE NORMAL (EX,STDX,X)
      SUM=0.0
      DO 5 I=1,26
      R=URN(1)
    5 SUM=SUM+R
      SUM=SUM/26.0
      X=STDX*17.61*(SUM-0.5)+EX
      RETURN
      END
```

NUMERICAL COMPUTATION OF VARIANCE OF $\hat{\Delta}$

```
      IMPLICIT REAL*8 (A-H,O-Z)
      BETA=.9
      DO 20 K=1,11
C     SMOOTHING CONSTANT
      ALPHA=1.-BETA
      ASUM=0.0
      WRITE(6,110) ALPHA
  110 FORMAT(//T10,'SMOOTHING CONSTANT =',F6.3)
C     APPROXIMATION OF THE INFINITE SUM
      DO 1 M=1,300
      I=M-1
      J=M
      T=4.0/(3.141*(1.0+BETA))
      P=ALPHA/(1.+BETA)
      D=BETA**(I+J)
      C=BETA**(J-I)
      A=C/BETA
    3 B=A*BETA
      RHO=(1.+BETA)/4.*(1.+P*(1.0-A**2+(1.+B)**2)+(1.-ALPHA*
     1A)**2-1.
      ZN=DSQRT(1.-RHO**2)
      ZP=-RHO/ZN
      SUM=T*(ZN-RHO*DATAN(ZP))*D
      ASUM=ASUM+SUM
      IF(SUM-1.0D-10) 1,1,2
    2 A=A*BETA
      D=D*BETA
      GO TO 3
    1 CONTINUE
      WRITE(6,102) SUM
  102 FORMAT(T10,'LAST SUM=',F20.12)
      X=ALPHA/(1.+BETA)**2*2.282/3.141
      Z=8.0/3.141*BETA/(1.0+BETA)**2
C     VARIANCE OF Δ̂
      VAR=X-Z+2.0*(1.0-BETA)**2*ASUM
C     STANDARD DEVIATION OF σ̃
      SDEV=DSQRT(3.141*(1.+BETA)*VAR)/2.0
      WRITE(6,101) VAR,SDEV
  101 FORMAT(T10,'VARIANCE OF MAD',F15.7,5X,'STANDART DEVIA
     1TION ',F15.7)
      BETA=BETA-.05
      IF(K.EQ.10) BETA=.2
   20 CONTINUE
      STOP
      END
```

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13. ABSTRACT

Prediction of demand and setting reorder levels by exponential smoothing and mean absolute deviation is studied for an inventory system. The relation between the standard deviation and mean absolute deviation of demand distribution is discussed. Asymptotic first and second moments of the exponentially-smoothed forecast errors (MAD) are derived. Results obtained from simulation of several normal systems indicated that the smoothing technique is inferior to the classical maximum likelihood estimation method.

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